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Multi-species asymmetric exclusion process in ordered sequential update

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Abstract. A multi-species generalization of the asymmetric simple exclusion process is studied with ordered sequential and sub-lattice parallel updating schemes. In this model, particles hop with their own specific probabilities to their rightmost empty site and fast particles overtake slow ones with a definite probability. Using the matrix product ansatz technique, we obtain the relevant algebra and study the uncorrelated stationary state of the model both for an open system and on a ring. A complete comparison between the physical results in these updates and those of random sequential introduced in Karimipour V 1999 A multi-species ASEP and its relation to traffic flow *Phys. Rev.* E **59** 205 and Karimipour V 1998 A multi-species ASEP, steady state and correlation functions on a periodic lattice *Preprint* cond-mat/9809193, is made.

1. Introduction

One-dimensional models of particles hopping in a preferred direction provide simple nontrivial realizations of systems out of thermal equilibrium [1–4]. In the past few years these systems have been extensively studied and now there is a relatively rich amount of results, both analytical and numerical, in the literature, (see [1, 4] and references therein). These types of models, which are examples of driven diffusive systems, exhibit interesting cooperative phenomena such as boundary-induced phase transition [5], spontaneous symmetry breaking [6, 7] and single-defect induced phase transitions [8–12, 24] which are absent in one-dimensional equilibrium systems.

A rather simple model which captures most of the mentioned features is the asymmetric simple exclusion process (ASEP) for which many analytical results have been obtained in one dimension [1, 4, 13]. Besides its usefulness in describing various problems such as the kinetics of biopolymerization, surface growth, Burger's equation and many others [4], ASEP has a natural interpretation as a model describing traffic flow on a one-lane road [14–16].

Derrida *et al* were the first to apply the matrix product ansatz technique (MPA) to the ASEP with open boundaries [17]. Since then, MPA has been applied to many other interesting stochastic models such as ASEP with a defect in the form of an additional particle with a different hopping rate [11], the two-species ASEP with oppositely charged particles moving in the same (opposite) directions [6, 12, 18] and many others. MPA has also been shown to be successful in describing disordered ASEP-like models. Evans [19] considered a model on a ring where each particle hops with its own specific rate to its right empty site if it is empty and stops otherwise (this model was simultaneously solved by Ferrari and Krug [19]). The model shows two phases. In low densities the hopping rate of the slowest particle determines the average velocities of particles (phase I). When the density of particles exceeds a critical value, it is then the total density which determines the average velocity and the slowest particle loses its predominant role (phase II). This model has many nice features both theoretically and idealistically but the possibility of exchanging between particles has not been considered.

Very recently in [20], a multi-species generalization of the ASEP has been proposed in which exchange processes among different species are implemented. In this model, there are p-species of particles present in an open chain with injection (extraction) of each species at boundaries. Each particle of i-type ($1 \leqslant i \leqslant p$) hops forward with rate v_i and can exchange its position with its right neighbour particle of j-type with rate $v_i - v_j$. The subtractive form of exchange rate allows only fast particles to exchange their positions with slow ones.

Most of the above-mentioned models have been defined in continuous time, where the master equation of the stochastic process can be written as a Schrödinger-like equation for a 'Hamiltonian' between nearest-neighbours [4, 22]. In contrast, one can use a discrete-time formulation of such random processes and adopt other types of updating schemes such as parallel, sub-parallel, forward and backward ordered sequential and particle ordered sequential (see [23] for a review). The MPA technique has been extended to a sub-lattice parallel updating scheme [25, 26] and, in the case of open boundary conditions, to an ordered sequential scheme [27, 28]. Although in traffic flow problems parallel updating is the most suitable scheme, few exact results are known [15, 29, 30].

In general, it is of prime interest to determine whether distinct updating schemes can produce different types of behaviour. The present analytical results show that with a change in the updating scheme of the model, general features and phase structure remain the same but the value of critical parameters may undergo some changes. In [23], Schreckenberg *et al* have considered the ASEP under three basic updating procedures. Similarities and differences have been fully discussed. Evans [29] has obtained analytical results in ordered and parallel updates for his model which was first solved in random sequential updating in [19]. He has demonstrated that the phase transition observed in [19] persists under parallel and ordered sequential updating.

In this paper, we aim to study the p-species model introduced in [20] under an ordered sequential update scheme and will show that the features observed in [20] are reproduced in ordered updating as well. Our results will be reduced to those of [23] when we set $p = 1$.

The organization of the paper is as follows. In section 2, we briefly explain the p-species ASEP with random sequential updating and then describe the MPA in backward ordered sequential updating for p-species ASEP and will obtain the related quadratic algebra. Section 3 contains the mapping of algebra of section 2 to that of [20] and includes the expressions for the currents and densities of each type of particle in the MPA approach. Section 4 is devoted to the one-dimensional representation of the quadratic algebra and the infinite limit ($p \to \infty$) of the number of species. In this limit, we use a continuum description of current–density diagrams of the model. In section 5, we consider the forward ordered updating and discuss the similarities and differences between forward and backward updating. In contrast to the usual ASEP where particle–hole symmetry allows for a map of result between forward and backward updating [23], here we do not have particle–hole symmetry and hence should separately consider the forward updating. At the end of this section, we discuss the intimate relationship between the sub-parallel scheme and the ordered sequential scheme [32]. Section 6 concludes the model with ordered updating on a closed ring. We obtain current–density diagrams for both backward and forward updating. The paper ends in section 7 with some concluding remarks.

2. The model

2.1. p-species ASEP in ordered sequential updating

In this section we first briefly describe the p-species ASEP introduced in [20]. This model consists of a one-dimensional open chain of length L. There are p species of particles and each site contains one particle at most. The dynamics of the model is exclusive and totally asymmetric to the right. Particles jump to their rightmost empty site, time is continuous and hopping of particles of type i $(1 \leqslant i \leqslant p)$ occurs with the rate v_i. To cast a model for describing traffic flow, the possibility of exchanging of two adjacent particles has been implemented, i.e. two neighbouring particles of types (j) and (i) swap their positions with rate $v_j - v_i$, $v_j > v_i$. This automatically forbids the exchange between low-speed and high-speed particles which is reminiscent of a one-way traffic flow where fast cars can overtake the slow ones. Denoting an i-type particle by A_i and a vacancy by ϕ, one can describe the dynamics by the following rules in the bulk:

$$A_i\phi \longrightarrow \phi A_i \qquad \text{with rate} \quad v_i \qquad (i = 1, \dots, p) \tag{1}$$

$$A_j A_i \longrightarrow A_i A_j \qquad \text{with rate} \quad v_j - v_i \qquad (j > i = 1, \dots, p) \tag{2}$$

where positiveness of the rates, imposes the following restrictions on the v_i:

$$v_1 \leqslant v_2 \leqslant v_3 \cdots \leqslant v_p. \tag{3}$$

To complete the definition of the process, one should consider the possibility of injection and extraction of particles at left and right boundaries. The injection (extraction) of particles of type i at left (right) boundary occurs with the rate α_i (β_i). This completes the definition of the model. Denoting the probability that at time t, the system contains particles of type τ_i ($\tau_i = 0$ refers to vacancy) at site i $(0 \leqslant \tau_i \leqslant p, 1 \leqslant i \leqslant L)$ by $P(\tau_1, \tau_2, \dots, \tau_L, t)$, one can write the stationary state $P_s(\tau_1, \tau_2, \dots, \tau_L)$ in the form of a matrix-product state (MPS)

$$P_s(\tau_1, \dots, \tau_L) \sim \langle W|D_{\tau_1} \dots D_{\tau_L}|V\rangle \tag{4}$$

in which D_{τ_i} is an ordinary matrix to be satisfied in some quadratic algebra induced by the dynamical rules of the model and the vectors $|V\rangle$, $\langle W|$ (reflecting the effect of the boundaries) act in some auxiliary space [31, 32]. Denoting D_0 by E, the quadratic algebra reads [20]

$$D_i E = \frac{1}{v_i} D_i + E \qquad (1 \leqslant i \leqslant p) \tag{5}$$

$$D_j D_i = \frac{1}{(v_i - v_j)}(v_i D_j - v_j D_i) \qquad (1 \leqslant i < j \leqslant p). \tag{6}$$

The vectors $|V\rangle$ and $\langle W|$ satisfy

$$D_i|V\rangle = \frac{v_i}{\beta_i}|V\rangle \tag{7}$$

$$\langle W|E = \langle W|\frac{v_i}{p\alpha_i}. \tag{8}$$

In [20] using MPA, an infinite-dimensional representation of the quadratic algebra is obtained but the form of current and density profiles could not been obtained by this infinite-dimensional representation. Instead, the simple case of one-dimensional representation was considered. Restricting the algebra to be one dimensional causes one to lose all the correlations, but many interesting features, such as a kind of Bose–Einstein condensation and boundary induced negative current [21], still appear, even in this simple uncorrelated case.

In what follows, we describe a p-species model under ordered sequential update. As stated in the introduction, in ordered sequential updating, time is discrete and the following events can happen in each timestep:

$$A_i\phi \longrightarrow \phi A_i \qquad \text{with probability} \quad v_i \qquad (i = 1, \ldots, p) \tag{9}$$

$$A_jA_i \longrightarrow A_iA_j \qquad \text{with probability} \quad f_{ji} \qquad (j > i = 1, \ldots, p). \tag{10}$$

We do not fix the form of f_{ji}: they will be fixed later. Particles are also injected (extracted) at the first (last) site with the probability α_i (β_i) and we denote the probability of the configuration $(\tau_1, \ldots, \tau_L)$ at the Nth timestep by $P(\tau_1, \ldots, \tau_L; N)$. We make a Hilbert space for each site of the lattice consisting of the basis vectors $\{|\tau\rangle, \tau = 0, \ldots, p\}$ where $|\tau\rangle$ denotes that the site contains a particle of type τ (vacancy is a particle of type 0). The total Hilbert space of the chain is the tensor product of these local spaces. With these constructions, the state of the system at the Nth timestep is defined to be $|P, N\rangle$ so that

$$P(\tau_1, \ldots, \tau_L; N) := \langle \tau_1, \ldots, \tau_L | P; N\rangle. \tag{11}$$

In ordered sequential updating, one can update the system from right to left or from left to right. In general these two schemes do not produce identical results and it is necessary to consider both of them separately. We first consider updating from right to left (backward). The state of the system at $(j+1)$th timestep is obtained from jth timestep as follows:

$$|P, j+1\rangle = T_{\leftarrow}|P, j\rangle \tag{12}$$

where $T_{\leftarrow}$ is

$$T_{\leftarrow} = L_1 T_{1,2} T_{2,3} \ldots T_{L-2,L-1} T_{L-1,L} R_L \tag{13}$$

and

$$L_1 = L \otimes 1 \otimes \cdots \otimes 1 \qquad R_L = 1 \otimes 1 \otimes \cdots \otimes R \tag{14}$$

$$T_{i,i+1} = 1 \otimes 1 \cdots \underbrace{1}_{i-1} \otimes T \otimes \underbrace{1}_{i+2} \cdots 1 \otimes 1. \tag{15}$$

According to (13), updating the state of the system in the next timestep consists of the $L+1$ consecutive sub-steps. First the site L is updated: if it is empty it is left unchanged, but if it contains a j-type particle ($1 \leqslant j \leqslant p$), this particle will be removed with the probability β_i from the site L of the chain, then the sites L and $L-1$ are updated by acting $T_{L-1,L}$ on $|\tau_{L-1}\rangle \otimes R|\tau_L\rangle$. The effect of $T_{L-1,L}$ is to update the site $L-1$ and L according to the stochastic rules (9) and (10). After updating all the links from right to left, one finally updates the first site: if it is occupied it is left unchanged, if it is empty then a particle of type i ($1 \leqslant i \leqslant p$) is injected with the probability α_i. This procedure defines one updating timestep. After many steps, one expects the system to reach its stationary state $|P_s\rangle$ which must not change under the action of $T_{\leftarrow}$ and therefore is an eigenvector of $T_{\leftarrow}$ with eigenvalue one:

$$|P_s\rangle = T_{\leftarrow}|P_s\rangle. \tag{16}$$

The explicit form of T, R and L can be written as

$$T = \sum_{i=1}^{p} v_i (E_{0i} \otimes E_{i0} - E_{ii} \otimes E_{00}) + \sum_{j>i=1}^{p} f_{ji} (E_{ij} \otimes E_{ji} - E_{jj} \otimes E_{ii}) + I \tag{17}$$

$$R = \sum_{i=1}^{p} \beta_i (E_{0i} - E_{ii}) + I \tag{18}$$

$$L = \sum_{i=1}^{p} \alpha_i (E_{i0} - E_{00}) + I. \tag{19}$$

Here the matrices E_{ij} act on the Hilbert space of one site and have the standard definition $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

2.2. MPA for ordered sequential scheme (backward)

In this section we introduce MPA for the p-species model with a backward ordered sequential updating scheme. As shown by Krebs and Sandow [31], the stationary state of a one-dimensional stochastic process with arbitrary nearest-neighbour interactions and random sequential update can always be written as a MPS [31]. In [32] Rajewsky and Schreckenberg have generalized this to ordered sequential and sub-parallel updating schemes which are intimately related to each other. Following [17, 23] we write

$$P_s(\tau_1, \dots, \tau_L) \sim \langle W|D_{\tau_1} \dots D_{\tau_L}|V\rangle \qquad (0 \leqslant \tau_i \leqslant p)$$

where the matrices $D_0, \dots, D_p$ and the vectors $|V\rangle$, $\langle W|$ are to be determined. Let us first write the above MPS in a more compact form by introducing two column matrices A and $\hat{A}$:

$$A = \begin{pmatrix} E \\ D_1 \\ D_2 \\ \vdots \\ D_p \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} \hat{E} \\ \hat{D}_1 \\ \hat{D}_2 \\ \vdots \\ \hat{D}_p \end{pmatrix}$$

(elements of A and $\hat{A}$ are usual matrices); so we formally write

$$|P_s\rangle = \frac{1}{Z_L}\langle\langle W|A \otimes A \otimes \cdots \otimes A|V\rangle\rangle \tag{20}$$

where the normalization constant Z_L is equal to $\langle W|C^L|V\rangle$ with $C = E + \sum_{i=1}^{p} D_i$. The bracket $\langle\langle \cdots \rangle\rangle$ indicates that the scalar product is taken in each entry of the vector $A \otimes A \cdots \otimes A$. One can easily check that (20) is indeed stationary, i.e. $T_{\leftarrow}|P_s\rangle = |P_s\rangle$, if the following conditions hold:

$$RA|V\rangle = \hat{A}|V\rangle \tag{21}$$

$$T(A \otimes \hat{A}) = \hat{A} \otimes A \tag{22}$$

$$\langle W|L\hat{A} = \langle W|A. \tag{23}$$

This simply means that a '*defect*' $\hat{A}$ is created in the beginning of an update at site $j = L$, which is then transferred through the chain until it reaches the left end where it disappears. Equations (17)–(19) and (21)–(23) lead to the following quadratic algebra in the bulk:

$$[D_i, \hat{D}_i] = [E, \hat{E}] = 0 \qquad i = 1, \dots, p \tag{24}$$

$$(1 - v_i)D_i\hat{E} - \hat{D}_i E = 0 \qquad i = 1, \dots, p \tag{25}$$

$$E\hat{D}_i + v_i D_i \hat{E} = \hat{E} D_i \qquad i = 1, \dots, p \tag{26}$$

$$f_{ji} D_j \hat{D}_i + D_i \hat{D}_j = \hat{D}_i D_j \qquad j > i = 1, \dots, p \tag{27}$$

$$(1 - f_{ji})D_j\hat{D}_i = \hat{D}_j D_i \qquad i > i = 1, \dots, p \tag{28}$$

and the following relations in the boundaries:

$$\langle W|\left(1 - \sum_{i=1}^{p} \alpha_i\right)\hat{E} = \langle W|E \tag{29}$$

$$\langle W|(\alpha_i \hat{E} + \hat{D}_i) = \langle W|D_i \qquad i = 1, \dots, p \tag{30}$$

$$\left(E + \sum_{i=1}^{p} \beta_i D_i\right)|V\rangle = \hat{E}|V\rangle \tag{31}$$

$$(1 - \beta_i)D_i|V\rangle = \hat{D}_i|V\rangle \qquad i = 1, \dots, p. \tag{32}$$

3. Mapping of the p-species ordered sequential algebra onto random sequential algebra

In this section we find a mapping between the algebra (24)–(32) and (5)–(8). This mapping for $p = 1$ (usual ASEP) was first performed in [33] where it was shown that, apart from some coefficients, ASEP in an open chain with either random or ordered update leads to the same quadratic algebra. Here we show that this correspondence holds for p-species ASEP as well. We first demand

$$\hat{E} = E + e \tag{33}$$

$$\hat{D}_i = D_i - d_i \qquad i = 1, \ldots, p \tag{34}$$

where e and d_i are c-numbers. Putting (33), (34) into (24)–(32) one arrives at

$$v_i D_i E = (1 - v_i) e D_i + d_i E \qquad i = 1, \ldots, p \tag{35}$$

$$f_{ji} D_j D_i = d_j D_i - d_i (1 - f_{ji}) D_j \qquad j > i = 1, \ldots, p \tag{36}$$

$$\langle W|E = \langle W|e \left(\frac{1}{\alpha} - 1\right) \tag{37}$$

$$D_i|V\rangle = \frac{d_i}{\beta_i}|V\rangle \qquad i = 1, \ldots, p \tag{38}$$

in which $\alpha = \sum_{i=1}^{p} \alpha_i$ and the following constraints have to be taken into account:

$$e = \sum_{i=1}^{p} d_i \qquad \alpha_i = \left(\frac{\alpha}{e}\right) d_i \qquad (i = 1, \ldots, p). \tag{39}$$

One should note that as soon as restricting the algebra (24)–(32) to the conditions (33), (34), the probabilities of injection are no longer free and are restricted by equation (39). Up to now the exchange probabilities f_{ji} have been free; however, we have not yet checked associativity of the algebra (35), (36). Taking into account the associativity fixes these exchange probabilities to be

$$f_{ji} = \frac{v_j - v_i}{1 - v_i} \qquad j > i = 1, \ldots, p. \tag{40}$$

Remark. *According to the discrete-time nature of updating procedure, f_{ji} are more precisely the conditional probabilities, i.e. they express the probability of exchanging between j and i-type particles provided that the i-type particle does not hop forward during the sub-timestep. Thus*

$$\mathrm{prob}(\ldots A_i A_j \ldots; N + 1| \ldots A_j A_i \ldots; N) \sim f_{ji}(1 - v_i) = v_j - v_i. \tag{41}$$

Therefore, we see that overtaking happens with a probability proportional to relative speed. With this requirement, equations (35)–(38) yield

$$v_i D_i E = (1 - v_i) e D_i + d_i E \qquad i = 1, \ldots, p \tag{42}$$

$$D_j D_i = \frac{1}{v_j - v_i}\{d_j (1 - v_i) D_i - d_i (1 - v_j) D_j\} \qquad j > i = 1, \ldots, p \tag{43}$$

$$\langle W|E = \langle W|e \left(\frac{1}{\alpha - 1}\right) \tag{44}$$

$$D_i|V\rangle = \frac{d_i}{\beta_i}|V\rangle \qquad i = 1, \ldots, p. \tag{45}$$

Equations (42)–(45) are the mapped algebra of p-species ASEP with backward ordered sequential updating onto random sequential updating. It can be easily verified that, similar to one-species ASEP [17], any representations of the algebra is either one- or infinite-dimensional. In the following, D_i and E are explicitly represented:

$$\tilde{E}=\begin{pmatrix} 0 & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . \\ . & 0 & 1 & 0 & . & . & . \\ . & . & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$\tilde{D}_i=\begin{pmatrix} \lambda_i & \frac{\lambda_i(1-v_i)}{v_i} & \lambda_i\frac{(1-v_i)^2}{{v_i}^2} & \lambda_i\frac{(1-v_i)^3}{{v_i}^3} & . & . & . \\ 0 & \frac{1}{v_i} & \frac{(1-v_i)}{v_i}\frac{1}{v_i} & \frac{(1-v_i)^2}{{v_i}^2}\frac{1}{v_i} & . & . & . \\ 0 & 0 & \frac{1}{v_i} & \frac{(1-v_i)}{v_i}\frac{1}{v_i} & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

with $\lambda_i = \frac{1}{(1+\eta)v_i-\eta}$ where η is a free parameter (we have a class of representations).

Using (45), we multiply both sides of (43) on $|V\rangle$ and obtain

$$v_j(1-\beta_i)-v_i(1-\beta_j)=\beta_j-\beta_i \qquad j>i=1,\ldots,p \tag{46}$$

which can be verified to have the solution

$$\beta_i=(1+\gamma)v_i-\gamma \qquad i=1,\ldots,p \tag{47}$$

in which γ is a free parameter. Equation (47) gives the β_i in terms of the v_i, i.e. given the hopping probability v_i, the extraction probabilities β_i are not free parameters any more. Requiring that all the probabilities be positive leads to the following condition on v_i:

$$\frac{\gamma}{\gamma+1}\leqslant v_1\leqslant v_2\cdots\leqslant v_p\leqslant 1 \qquad \gamma\epsilon[0,\infty[. \tag{48}$$

We conclude this section with formulae for the current operators. In contrast to random sequential updating where currents are local, i.e. caused by, at most, a single hopping of particles, in the ordered sequential updating the currents are highly nonlocal, which is to say that they can have many hopping sources according to the multiplicative nature of transition matrix $T_{\leftarrow}$. In ordered sequential updating, the mean current in the Nth timestep through the site k is defined by

$$\langle n_k^{(i)}\rangle_{N+1}-\langle n_k^{(i)}\rangle_N=\langle J_{k-1,k}^{(i)}\rangle_N-\langle J_{k,k+1}^{(i)}\rangle_N. \tag{49}$$

Our attention is concentrated on the stationary state and thus the limit $N\longrightarrow\infty$ has to be considered. Upon introducing a bra vector

$$\langle S| := \sum_{\tau_1,\ldots,\tau_N}\langle\tau_1,\ldots,\tau_L|$$

the l.h.s. of (49) can be written as

$$\langle S|n_k^{(i)}T_{\leftarrow}T_{\leftarrow}^N|P(0)\rangle-\langle S|n_k^{(i)}T_{\leftarrow}^N|P(0)\rangle \tag{50}$$

which in turn yields

$$\langle n_k^{(i)}\rangle_{N+1}-\langle n_k^{(i)}\rangle_N=\langle S|[n_k^{(i)},T_{\leftarrow}]|P_s\rangle. \tag{51}$$

We have used the fact that $\langle S|T_{\leftarrow}=\langle S|$, which is justified if $T_{\leftarrow}$ is the transfer matrix of a stochastic process. Evaluating the commutator in (51), everything is expressed in stationary state expectation values of densities which, using MPS (20), finally leads to the expression for the current of i-type particles from the site $k-1$ to k

$$\langle J_{k-1,k}^{(i)}\rangle_{\leftarrow} = \frac{\langle W|C^{k-2}J^{(i)}C^{L-k}|V\rangle}{\langle W|C^{L}|V\rangle} \tag{52}$$

in which

$$J^{(i)} = v_i D_i \hat{E} + \sum_{j>i}^{p} \frac{v_i - v_j}{1-v_j} D_i \hat{D}_j - \sum_{j>i}^{p} \frac{v_j - v_i}{1-v_i} D_j \hat{D}_i \tag{53}$$

and

$$C = E + \sum_{i=1}^{p} D_i. \tag{54}$$

The first term in (53) is due to hopping of the i-type particles, the second term corresponds to the exchanges between an i-type and all the particles with lower hopping probabilities, and finally the last term expresses the exchanging between all the particles with higher hopping probabilities and the i-type particle.

Using (33), (34) and the bulk algebra (42) and (43), one easily concludes that

$$J^{(i)} = d_i C. \tag{55}$$

Thus the current and density of i-type particles through (at) site k are respectively given by

$$\langle J_{k}^{(i)}\rangle_{\leftarrow} = d_i \frac{\langle W|C^{L-1}|V\rangle}{\langle W|C^{L}|V\rangle} \tag{56}$$

$$\langle n_{k}^{(i)}\rangle_{\leftarrow} = \frac{\langle W|C^{k-1}D_i C^{L-k}|V\rangle}{\langle W|C^{L}|V\rangle}. \tag{57}$$

Therefore, all the currents are proportional to the average current $J_{\leftarrow}$, whereas $J_{\leftarrow}$ has a nontrivial dependence on hopping probabilities. The next section is devoted to the one-dimensional representation of the algebra (42)–(45). This case corresponds to the steady state characterized by a Bernoulli measure. In spite of its simplicity, still some interesting features survive in one-dimensional representation.

4. One-dimensional representation and infinite-species limit

4.1. One-dimensional representation

The simplest representation of the algebra (42)–(45) is to take the dimension of the matrices to be one. For later convenience, let us replace all D_i by $\frac{D_i}{p}$ where p is the number of species. Denoting $\frac{D_i}{p}$ and E by c-numbers, $\frac{\mathcal{D}_i}{p}$ and $\mathcal{E}$ respectively, from equations (44) and (45) we have

$$\mathcal{D}_i = \frac{pd_i}{(1+\gamma)v_i - \gamma} \qquad \mathcal{E} = e\left(\frac{1}{\alpha} - 1\right). \tag{58}$$

Putting these numbers in (42) leads to

$$v_i = 1 \qquad \text{or} \qquad \frac{1}{\alpha} - \frac{1}{\gamma} = 1. \tag{59}$$

The case $v_i = 1$ corresponds to the ordinary 1-species ASEP which has been extensively studied. Using (47), the second condition can be written as

$$(1-\alpha)(1-\bar{\beta}) = (1-e) \tag{60}$$

in which

$$\alpha = \sum_{i=1}^{p} \alpha_i \qquad \bar{\beta} = \sum_{i=1}^{p} \frac{\beta_i}{p}. \tag{61}$$

α is the total probability of injection of particles (note that α should be less than one) and $\bar{\beta}$ is the average probability of extraction of particles. In the special case of 1-species, (60) reduces to

$$(1-\alpha_1)(1-\beta_1) = 1 - d_1.$$

Comparing this with the usual ASEP [33] in which the condition for one-dimensional representation reads to be $(1-\alpha)(1-\beta) = 1-p$ (p is the hopping probability) causes us to take e as the average probability of hopping, i.e. $e = \sum_{i=1}^{p} \frac{v_i}{p}$. So a natural choice for d_i would be to use $\frac{v_i}{p}$. Now, α_i is proportional to $\frac{1}{p}$ and this guarantees the convergence of the sum $\alpha = \sum_{i=1}^{p} \alpha_i$ in the large-p limit. On the other hand, β_i are no longer proportional to $\frac{1}{p}$ and the appearance of the factor $\frac{1}{p}$ in $\bar{\beta}$ is necessary to make $\bar{\beta}$ convergent in the large-p limit. In one-dimensional representation, the hopping probabilities are restricted to

$$\alpha \leqslant v_1 \leqslant v_2 \leqslant v_3 \cdots \leqslant v_p \leqslant 1. \tag{62}$$

Within one-dimensional representation, the stationary state is uncorrelated and is given by $|P_s\rangle = |\rho\rangle^{\otimes L}$ where

$$|\rho\rangle = \frac{1}{c}\begin{pmatrix} \mathcal{E} \\ \frac{\mathcal{D}_1}{p} \\ \frac{\mathcal{D}_2}{p} \\ \vdots \\ \frac{\mathcal{D}_p}{p} \end{pmatrix} \qquad c = \mathcal{E} + \frac{1}{p}(\mathcal{D}_1 + \mathcal{D}_2 \ldots \mathcal{D}_p) \equiv \mathcal{E} + \frac{1}{p}\mathcal{D}. \tag{63}$$

The density and the current of i-type particles are all site independent and are respectively given by equations (57) and (56):

$$\rho_{\leftarrow}(\alpha, i) = \frac{\frac{\mathcal{D}_i}{p}}{e(\frac{1}{\alpha}-1) + \frac{\mathcal{D}}{p}} \qquad J_{\leftarrow}(\alpha, i) = \frac{\frac{v_i}{p}}{e(\frac{1}{\alpha}-1) + \frac{\mathcal{D}}{p}}. \tag{64}$$

One can define total density and total current by summing over all kind of species and find

$$\rho_{\leftarrow}(\alpha) = \frac{\frac{\mathcal{D}}{p}}{e(\frac{1}{\alpha}-1) + \frac{\mathcal{D}}{p}} \qquad J_{\leftarrow}(\alpha) = \frac{e}{e(\frac{1}{\alpha}-1) + \frac{\mathcal{D}}{p}}. \tag{65}$$

4.2. Infinite-species limit

At this stage we consider the limit $p \to \infty$, and assume that the hopping probabilities of particles are chosen from a continuous distribution $P(v)$. Discrete quantities $\frac{1}{p}F(i)$ are transformed into $f(v)P(v)$ and sums into integrals. Equations (64) and (65) then take the form

$$\rho_{\leftarrow}(\alpha, v) = \frac{\mathcal{D}(\alpha, v)P(v)}{e(\frac{1}{\alpha}-1) + \mathcal{D}(\alpha)} \qquad J_{\leftarrow}(\alpha, v) = \frac{vP(v)}{e(\frac{1}{\alpha}-1) + \mathcal{D}(\alpha)} \tag{66}$$

$$\rho_{\leftarrow}(\alpha) = \frac{\mathcal{D}(\alpha)}{e(\frac{1}{\alpha}-1) + \mathcal{D}(\alpha)} \qquad J_{\leftarrow}(\alpha) = \frac{e}{e(\frac{1}{\alpha}-1) + \mathcal{D}(\alpha)} \tag{67}$$

where

$$\mathcal{D}(\alpha, v) = \frac{(1-\alpha)v}{v-\alpha} \qquad \text{and} \qquad \mathcal{D}(\alpha) = (1-\alpha)\int_\alpha^1 \frac{v}{v-\alpha} P(v)\,\mathrm{d}v. \tag{68}$$

Although one has many choices for $P(v)$, we first take the following [19]. It has the merit that $\mathcal{D}(\alpha)$ can be analytically evaluated.

$$P_1(v) = \frac{(m+1)}{(1-\alpha)^{m+1}}(v-\alpha)^m \qquad m \geqslant 0. \tag{69}$$

This is a normalized distribution that vanishes with some positive power in low-velocities and increases up to $v = 1$. The average hopping probability e is found to be

$$e = \int_\alpha^1 vP_1(v)\,\mathrm{d}v = \frac{(m+1)}{(m+2)}(1-\alpha) + \alpha.$$

Expressing m in terms of e and α we have

$$m = \frac{2e-\alpha-1}{1-e}. \tag{70}$$

For m to be positive, (70) implies $(e, \alpha \leqslant 1)$

$$2e - \alpha - 1 \geqslant 0. \tag{71}$$

We first study the current–density relationship for a fixed hopping probability, e. In order to do this, we evaluate $\mathcal{D}(\alpha)$ with (68) and replace m from (70):

$$J_\leftarrow(\alpha, e) = \frac{e}{e(\frac{1}{\alpha}-1) + \frac{2e-1-\alpha e}{2e-1-\alpha}} \tag{72}$$

$$\rho_\leftarrow(\alpha, e) = \frac{\frac{2e-1-\alpha e}{2e-1-\alpha}}{e(\frac{1}{\alpha}-1) + \frac{2e-1-\alpha e}{2e-1-\alpha}}. \tag{73}$$

The above expressions give the total current and total density in terms of two control parameters, namely the total arrival probability α and the average hopping probability e. We now eliminate α between $J_\leftarrow$ and $\rho_\leftarrow$ numerically, which then gives the current density diagram. This diagram is shown in figure 1 for two values of e.

Remark. *Total current $J_\leftarrow$ and total density $\rho_\leftarrow$ are, in general, functions of three control parameters e, α and m. Recalling that e is the average hopping probability, α is the total rate of injection and m determines the shape of hopping distribution function, equation (70) implies that only two parameters are independent. There is a one-to-one correspondence between the two-dimensional parameter space defined by the surface (70) and the current–density space. $J_\leftarrow$ versus $\rho_\leftarrow$ in figure 1 corresponds to the intersection of planes e = constant, with the surface defined by (70). We can instead look at the intersection of α = constant planes with the surface and find the corresponding curves in the $J_\leftarrow$–$\rho_\leftarrow$ plane. This is performed by eliminating e between equations (72) and (73). Figure 2 shows these diagrams for some values of α.*

Finally, we consider the curves of constant m in the $J_\leftarrow$–$\rho_\leftarrow$ plane. To obtain these curves, one should express $J_\leftarrow$ and $\rho_\leftarrow$ in terms of α and m as follows:

$$J_\leftarrow(\alpha, m) = \frac{\alpha(\alpha+m+1)}{(\alpha+m+1)(1-\alpha) + \alpha(m+2)(1+\frac{\alpha}{m})} \tag{74}$$

$$\rho_\leftarrow(\alpha, m) = \frac{\alpha(m+2)(1+\frac{\alpha}{m})}{(\alpha+m+1)(1-\alpha) + \alpha(m+2)(1+\frac{\alpha}{m})}. \tag{75}$$

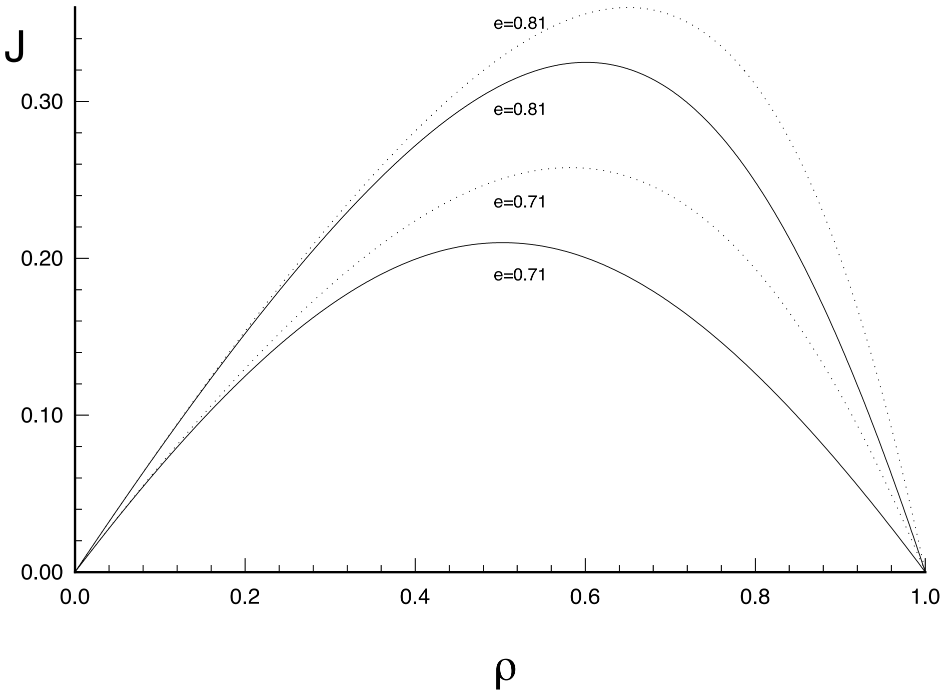

Figure 1. The current versus the density for different values of e in backward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

Eliminating α between $\rho_{\leftarrow}(\alpha, m)$ and $J_{\leftarrow}(\alpha, m)$ would give us the current–density diagrams for a fixed value of m. Figure 3 shows these diagrams for some values of m. As can be seen, the current does not vanish at $\rho_{\leftarrow} = 1$. This can be explained by noticing that, although at $\rho_{\leftarrow} = 1$, the chain is completely filled, we still have current via exchange processes. At $\rho_{\leftarrow} = 1$, the more m decreases, the more $J_{\leftarrow}$ approaches zero.

Using (72) and (74), we can also look at the behaviour of current itself as a function of control parameters. In figures 4 and 5, we show the dependence of $J_{\leftarrow}$ on α, e for some fixed values of e and α. Note that for each α, there is a lower limit of e which can be obtained through equation (70).

Our second choice of velocity distribution function is the following:

$$P_2(v) = \frac{(m+1)(m+2)}{(1-\alpha)^{m+2}}(v-\alpha)^m(1-v) \qquad m \geqslant 0. \tag{76}$$

It vanishes at $v = \alpha$, $v = 1$ and has a maximum at $v_{max} = \frac{m+\alpha}{m+1}$. If m increases, v_{max} approaches to one and if m decreases to zero , v_{max} approaches to α. Inserting $P_2(v)$ into (39) we find

$$m = \frac{3e - 2\alpha - 1}{1 - e}. \tag{77}$$

Using (67), (68) and (77), we express $J_{\leftarrow}$ and $\rho_{\leftarrow}$ in terms of e, α and α, m:

$$J_{\leftarrow}(\alpha, e) = \frac{e\alpha(2\alpha + 1 - 3e)}{e(1-\alpha)(2\alpha - 3e + 1) + \alpha(2\alpha e - 3e + 1)} \tag{78}$$

$$\rho_{\leftarrow}(\alpha, e) = \frac{\alpha(2\alpha e + 1 - 3e)}{e(1-\alpha)(2\alpha - 3e + 1) + \alpha(2\alpha e - 3e + 1)} \tag{79}$$

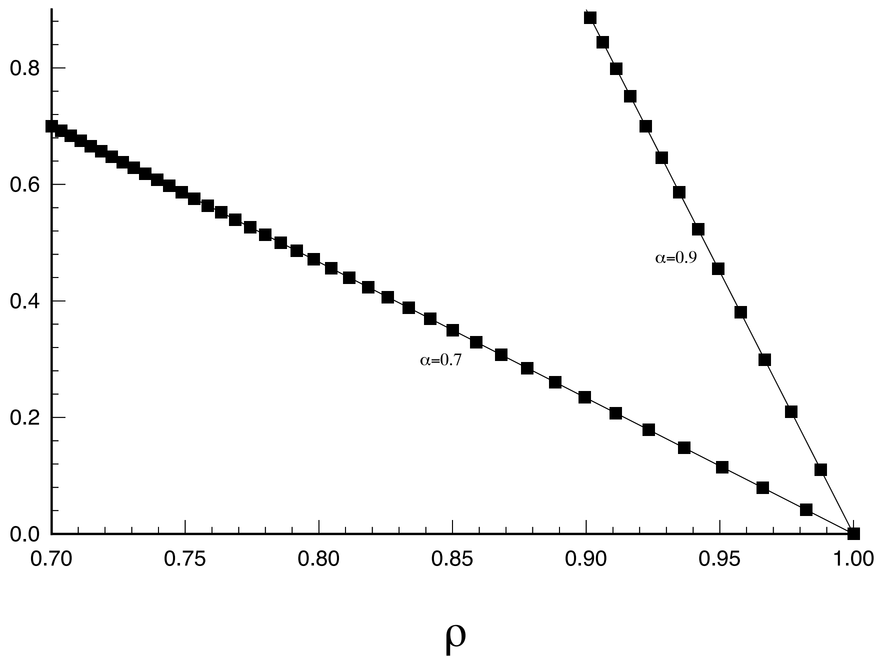

Figure 2. The current versus the density for different values of α in backward updating. Continuous lines refer to $P_1(v)$ and filled squares refer to $P_2(v)$.

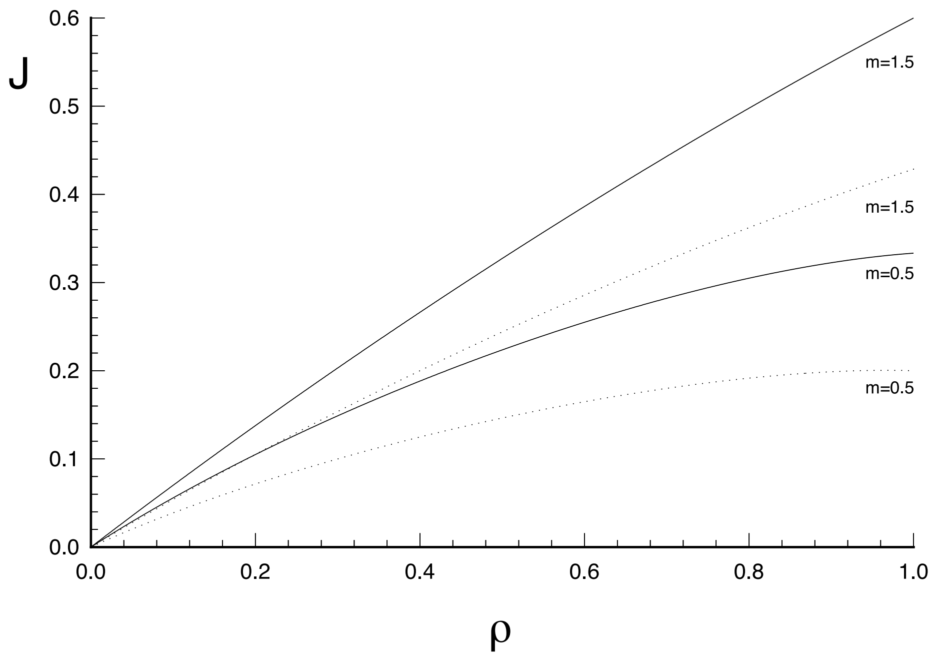

Figure 3. The current versus the density for different values of m in backward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

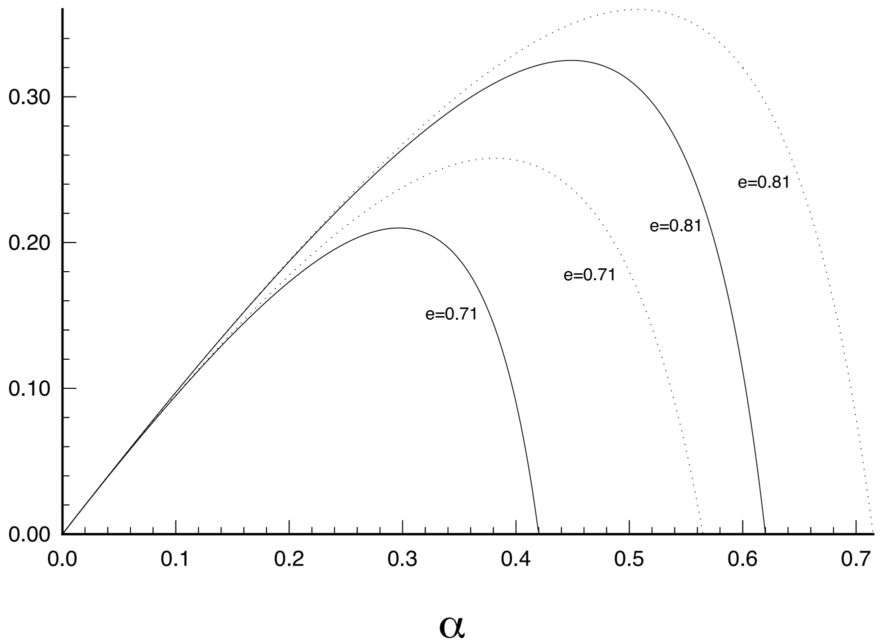

Figure 4. The current versus the arrival probability of particles for different values of e in backward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

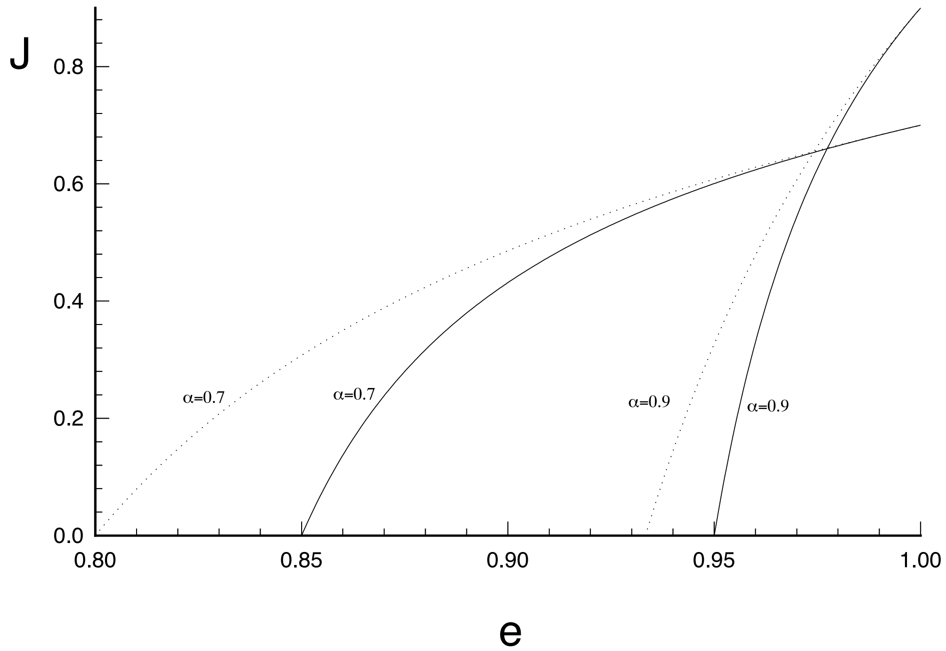

Figure 5. The current versus the total probability of hopping for different values of α in backward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

$$J_{\leftarrow}(\alpha, m) = \frac{\alpha(2\alpha + m + 1)}{(1-\alpha)(2\alpha + m + 1) + \alpha(m+3)(\frac{2\alpha}{m} + 1)} \tag{80}$$

$$\rho_{\leftarrow}(\alpha, m) = \frac{\alpha(m+3)(\frac{2\alpha}{m} + 1)}{(1-\alpha)(2\alpha + m + 1) + \alpha(m+3)(\frac{2\alpha}{m} + 1)}. \tag{81}$$

We now eliminate α between $J_{\leftarrow}(\alpha, e)$ and $\rho_{\leftarrow}(\alpha, e)$ which leads to current–density diagrams for fixed values of e. Dotted curves in figure 1 shows these diagrams for the same values of e.

Similar to $P_1(v)$, we can consider the current–density diagrams corresponding to constant α and m. These diagrams are depicted in figures 2 and 3, respectively. Dependence of $J_{\leftarrow}$ on α and e for $P_2(v)$ is also shown in figures 4 and 5 by dotted curves. Note that, in figure 5, the curves obtained from $P_1(v)$ asymptotically approach those of $P_2(v)$.

Here, we would like to discuss a feature of the infinite-species limit which is somehow reminiscent of Bose–Einstein condensation [19]. Equation (68) implies that the density of particles with speed v is proportional to $\frac{vP(v)}{v-\alpha}$. Taking (69), (76) for $P(v)$ we have

$$\rho(v) \sim v(v-\alpha)^{m-1}. \tag{82}$$

Recalling that α is the minimum speed of particles, equation (82) shows two different kinds of behaviour depending on whether $m > 1$ or $m < 1$.

(I) If $m - 1 > 0$ then $\rho(v) \to 0$ for $v \to \alpha$ which means that density of low-speed particles is small, i.e. most of the particles move with rather high speed.

(II) If $m - 1 < 0$ then $\rho(v) \to \infty$ for $v \to \alpha$. In contrast to case (I), here the density of low-speed particles is large and most of the particles move with low speed, which can be interpreted as appearance of the traffic jam phase.

5. *p*-species ASEP with forward updating

5.1. Formulation

As discussed in the introduction and section 2, instead of right to left (backward) updating, one can change the direction of updating and start from the first site of the chain (forward updating), updating from the left to the right in the same manner as for backward updating. Most of the steps are similar to backward updating and we only present the results. The transfer matrix takes the following form:

$$T_{\rightarrow} = R_L T_{L-1,L} \ldots T_{1,2} L_1. \tag{83}$$

All the matrices are the same as in (17)–(19). The MPS for the steady state is written as [23]

$$|P_s\rangle_{\rightarrow} = \langle\langle W|\hat{A} \otimes \hat{A} \otimes \cdots \otimes \hat{A}|V\rangle\rangle. \tag{84}$$

Assuming A and $\hat{A}$ to satisfy the same algebra, (21)–(23), makes $|P_s\rangle_{\rightarrow}$ a stationary state, i.e. $T_{\rightarrow}|P_s\rangle_{\rightarrow} = |P_s\rangle_{\rightarrow}$. Here at first site $i = 1$ a '*defect*' A is created, then transmitted forward until it reaches the last site $i = L$, where it disappears. Next, we consider formulae for the currents and densities. Here the situation is quite different and the difference between forward and backward updating reveals itself. The definition of currents is taken from (49)–(51) and $T_{\leftarrow}$ is replaced with (83). The mean current of i-type particles through site k is found to be

$$\langle J^{(i)}_{k-1,k}\rangle_{\rightarrow} = \frac{\langle W|\hat{C}^{k-2} J^{(i)} \hat{C}^{L-k}|V\rangle}{\langle W|\hat{C}^L|V\rangle} \tag{85}$$

where $J^{(i)}$ is the same as equation (53), and $\hat{C} = \hat{E} + \sum_{i=1}^{p} \hat{D}_i$. We again demand that $\hat{E}$ and $\hat{D}_i$ satisfy equations (33), (34) which in turn lets us revisit equations (42)–(45) and thus we have

$$J^{(i)} = d_i C \tag{86}$$

$$\hat{C} = C. \tag{87}$$

Putting (86), (87) in (85) yields

$$\langle J^{(i)} \rangle_{\to} = d_i \frac{\langle W|C^{L-2}|V\rangle}{\langle W|C^L|V\rangle}. \tag{88}$$

Also, one can write the mean density of i-type particles at site k:

$$\langle n_k^{(i)} \rangle_{\to} = \frac{\langle W|C^{k-1}(D_i - v_i)C^{L-k}|V\rangle}{\langle W|C^L|V\rangle}. \tag{89}$$

5.2. One-dimensional representation and infinite number of species limit in forward updating

We now scale all D_i by a $\frac{1}{p}$ factor and take $\frac{D_i}{p}$ and E to be c-numbers. Similarly to backward updating, we denote them by $\frac{\mathcal{D}_i}{p}$ and $\mathcal{E}$, respectively, and the equations (58)–(61) remain the same. In one-dimensional representation, the densities and the currents of i-type particles are all site independent and are respectively given by

$$\rho_{\to}(\alpha, i) = \frac{\left(\frac{\mathcal{D}_i}{p} - \frac{v_i}{p}\right)}{e\left(\frac{1}{\alpha} - 1\right) + \frac{\mathcal{D}}{p}} \qquad J_{\to}(\alpha, i) = \frac{\frac{v_i}{p}}{e\left(\frac{1}{\alpha} - 1\right) + \frac{\mathcal{D}}{p}}. \tag{90}$$

Comparing the above equations with their counterparts in backward updating, we see that currents do not change but forward density undergoes the following modifications:

$$J_{\to}(\alpha, i) = J_{\leftarrow}(\alpha, i) = J(\alpha, i) \qquad \rho_{\to}(\alpha, i) = \rho_{\leftarrow}(\alpha, i) - J(\alpha, i). \tag{91}$$

The above relations express the difference between forward and backward updating. A similar relation between backward and forward densities is seen in [23]. We again define the total density and current by summing over densities and currents of all kind of species:

$$J_{\to}(\alpha) = J_{\leftarrow}(\alpha) = \frac{e}{e\left(\frac{1}{\alpha} - 1\right) + \frac{\mathcal{D}}{p}} \qquad \rho_{\to}(\alpha) = \rho_{\leftarrow}(\alpha) - J(\alpha). \tag{92}$$

Now we take the limit of $p \to \infty$. Adopting the same distribution functions $P_1(v)$, $P_2(v)$ and using (92), one easily can obtain $J_{\to}$ and $\rho_{\to}$ as functions of e, α and m, both for $P_1(v)$ and $P_2(v)$. Similar to the backward scheme, the corresponding current–density diagrams can be obtained by eliminating one of the control parameters. These diagrams are shown in figures 6–8.

Remark. *Surprisingly, as can be seen in figure 7, when $\rho_{\to}$ goes to zero, $J_{\to}$ does not vanish. This is an exclusive effect appearing only in forward updating.*

It can be explained by noting that, according to the equations (92), (78) and (79), $\rho_{\to} = 0$ yields $e = 1$. This means that we can only have one type of particles in the system which deterministically hops with unit probability.

When the lattice is completely empty, i.e. $\rho_{\to} = 0$, in the first site a particle is injected with the probability α, then, according to the multiplicative nature of the transition matrix, is transferred through the lattice. Hence one has a non-zero current.

In general, the value of $J_{\to}$ at $\rho_{\to}$ is equal to α and this point refers to the point $(m = \infty, e = 1, \alpha)$ in parameter space.

We would like to end this section with some remarks on the sub-parallel updating scheme. In fact, as stated in section 1, there are few exact results in parallel updating. The root of

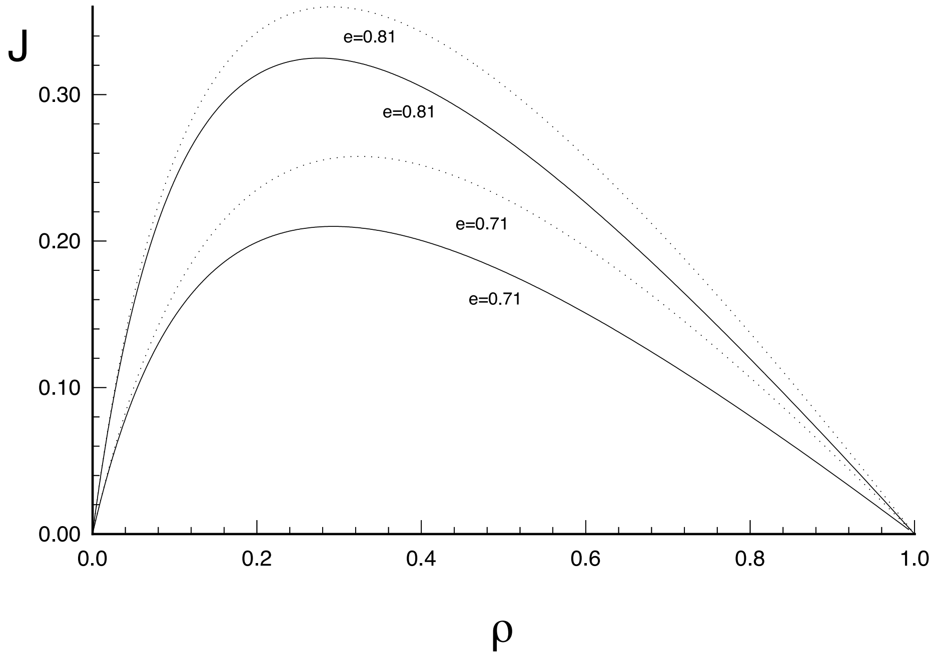

Figure 6. The current versus the density for different values of e in forward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

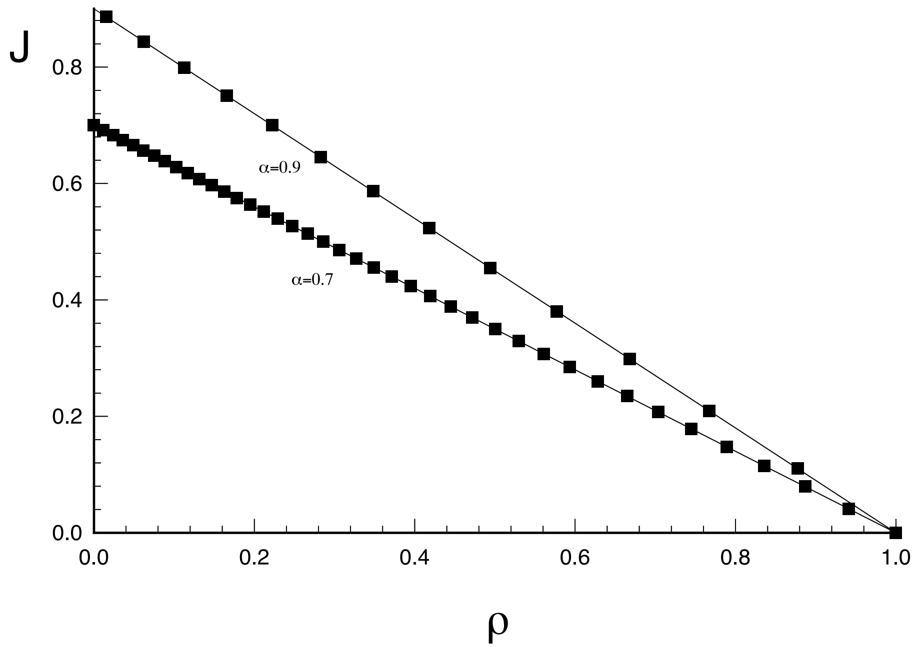

Figure 7. The current versus the density for different values of α in forward updating. Continuous lines refer to $P_1(v)$ and filled squares refer to $P_2(v)$.

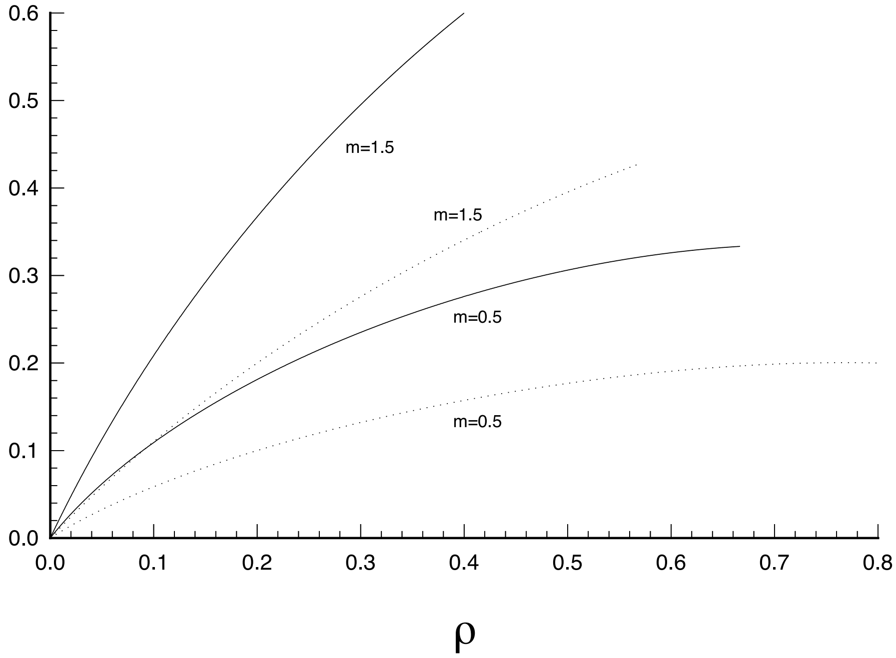

Figure 8. The current versus the density for different values of m in forward updating. Continuous curves refer to $P_1(v)$ and dotted curves refer to $P_2(v)$.

this difficulty is the non-local nature of transfer matrix which unlike to the ordered sequential updating, can not be written as a product of local transfer matrices. A simpler case is to consider a sub-parallel updating scheme [24]. In this scheme, one proceeds with two half-timesteps. In the first half, one updates the first site, last site and all pairs (τ_i, τ_{i+1}) with an even i (L is taken to be even). Then in the second half-timestep, one updates all pairs (τ_i, τ_{i+1}) with i odd. Thus the transfer matrix is

$$T_{sp} = T_{sp}^{(2)} T_{sp}^{(1)} \tag{93}$$

with

$$T_{sp}^{(1)} = L_1 T_{2,3} T_{4,5} \ldots T_{L-2,L-1} R_L \tag{94}$$

$$T_{sp}^{(2)} = T_{1,2} T_{3,4} \ldots T_{L-1,L}. \tag{95}$$

Defining MPS for sub-parallel updating as follows [25]:

$$|P_s\rangle_{sp} = \langle\langle W|\hat{A} \otimes A \otimes \hat{A} \otimes \cdots \hat{A} \otimes A|V\rangle\rangle. \tag{96}$$

It can be verified that $T_{sp}|P_s\rangle_{sp} = |P_s\rangle_{sp}$ provided that equations (21)–(23) are satisfied.

It is shown in [32] that sub-parallel and ordered sequential updating schemes are intimately related to each other. It is proved that, in general, the following correspondence exists:

$$\langle n_k^{(i)}\rangle_{sp} = \begin{cases} \langle n_k^{(i)}\rangle_{\rightarrow} & k \text{ odd} \\ \langle n_k^{(i)}\rangle_{\leftarrow} & k \text{ even} \end{cases} \tag{97}$$

$$\langle n_k^{(i)} n_l^{(j)}\rangle_{sp} = \begin{cases} \langle n_k^{(i)} n_l^{(j)}\rangle_{\rightarrow} & k, l \text{ odd} \\ \langle n_k^{(i)} n_l^{(j)}\rangle_{\leftarrow} & k, l \text{ even} \end{cases} \tag{98}$$

where k and l refer to the lattice sites and i and j refer to the state of the site. Using this general correspondence, we obtain the density profile of p-species ASEP under sub-parallel updating (one-dimensional representation)

$$\langle n_k^{(i)}\rangle_{sp} = \frac{\frac{\mathcal{D}_i}{p}}{e(\frac{1}{\alpha}-1)+\frac{\mathcal{D}}{p}} \qquad k = \text{even} \tag{99}$$

$$\langle n_k^{(i)}\rangle_{sp} = \frac{\frac{\mathcal{D}_i}{p}-\frac{v_i}{p}}{e(\frac{1}{\alpha}-1)+\frac{\mathcal{D}}{p}} \qquad k = \text{odd}. \tag{100}$$

6. p-species ASEP with ordered updating on a ring

In this section we consider the p-species ASEP on a closed ring of N sites. We work in a canonical ensemble in which the number of each species (i) is fixed to be m_i and take the total number of particles to be M, i.e. $\sum_{i=1}^{p} m_i = M$.

The periodic system can be described by a one-dimensional representation of the bulk algebra (24)–(28). In this case, the bulk algebra reduces to the following equations:

$$(1-v_i)d_i\hat{e} = \hat{d}_i e \tag{101}$$

$$\left(\frac{1-v_j}{1-v_i}\right)d_j\hat{d}_i = \hat{d}_j d_i. \tag{102}$$

The above equations yield

$$\hat{d}_j = \frac{\hat{e}}{e}(1-v_j)d_j. \tag{103}$$

Here d_i and $\hat{d}_i$ correspond to one-dimensional representations of D_i and $\hat{D}_i$ (not to be confused with those introduced in (34)). Using (53) and (57) we obtain the following forms for the density and the current of i-type particles:

$$\rho_{\leftarrow}^{(i)} = \frac{d_i}{e+\sum_i d_i} \qquad J_{\leftarrow}^{(i)} = \hat{e}\left(v_i d_i + \frac{1}{e}\left[v_i d_i \sum_j d_j - d_i \sum_j d_j v_j\right]\right). \tag{104}$$

Summing over i, we obtain the total current and density

$$\rho_{\leftarrow} = \frac{\sum_i d_i}{e+\sum_i d_i} \qquad J_{\leftarrow} = \hat{e}\sum_i v_i d_i. \tag{105}$$

Defining the population averaged velocity $\langle v\rangle$ as follows:

$$\langle v\rangle = \frac{\sum_i m_i v_i}{\sum_i m_i} \tag{106}$$

and rescaling the d_i and e so that

$$e + \sum_i d_i = \hat{e} + \sum_i \hat{d}_i = 1 \tag{107}$$

we arrive at

$$J_{\leftarrow} = \frac{\langle v\rangle\rho_{\leftarrow}(1-\rho_{\leftarrow})}{1-\langle v\rangle\rho_{\leftarrow}} \tag{108}$$

which is the current–density relation of p-species ASEP on a ring with backward updating. Comparing it with the usual ASEP on a ring with backward updating in [23], we see that they both have the same form. In the p-species model, $\langle v\rangle$ plays the role of hopping probability

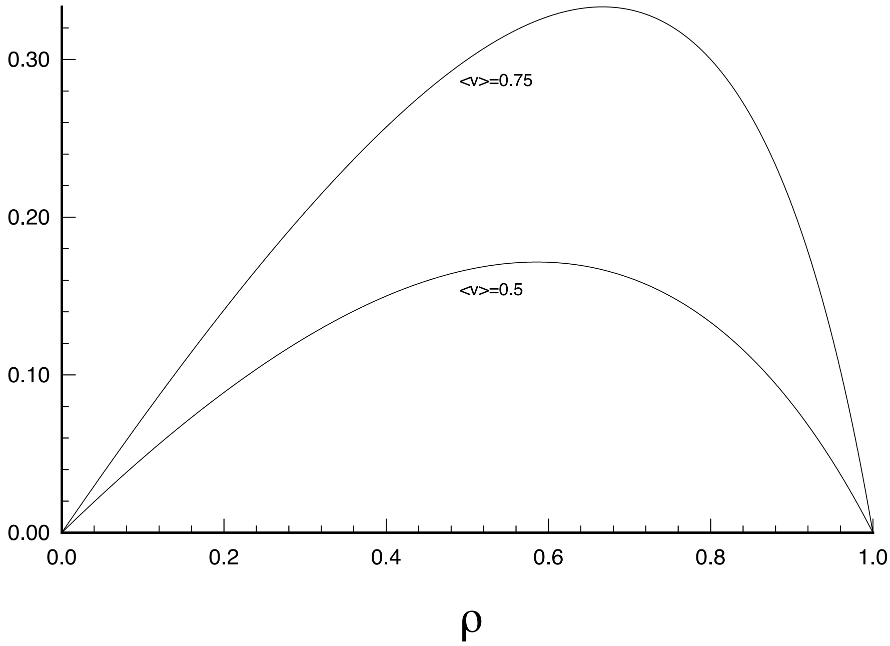

Figure 9. The current versus the density for different values of $\langle v\rangle$ in backward updating.

in the usual ASEP. Figure 9 shows $J_\leftarrow$ versus $\rho_\leftarrow$ for different values of $\langle v\rangle$. The maximum current occurs at

$$\rho_\leftarrow^{max}(\langle v\rangle) = \frac{1-(1-(\langle v\rangle))^{\frac{1}{2}})}{\langle v\rangle} \geqslant \frac{1}{2}. \tag{109}$$

We now consider the forward updating. Note that since we do not have particle–hole symmetry, the current–density relation in forward updating cannot be obtained from the one in backward updating and should be considered separately. In forward updating we have

$$\rho_\rightarrow^{(i)} = \frac{\hat{d}_i}{\hat{e} + \sum_i \hat{d}_i} \qquad J_\leftarrow^{(i)} = J_\rightarrow^{(i)}. \tag{110}$$

Using (101)–(103) and (107), after straightforward calculations, we find

$$J_\rightarrow = \frac{(1-\rho_\rightarrow)\rho_\rightarrow\langle\frac{v}{1-v}\rangle}{1+\rho_\rightarrow\langle\frac{v}{1-v}\rangle} \tag{111}$$

where

$$\left\langle\frac{v}{1-v}\right\rangle = \frac{\sum_i \frac{v_i}{1-v_i}m_i}{\sum_i m_i}. \tag{112}$$

If we now take $p=1$, $\langle\frac{v}{1-v}\rangle$ will reduce to $\frac{v_1}{1-v_1}$ and (111) takes the following form:

$$J_\rightarrow = \frac{v_1\rho_\rightarrow(1-\rho_\rightarrow)}{1-v_1\rho_\rightarrow} \tag{113}$$

and it can be verified that the particle–hole symmetry is restored [23], i.e. (113) is obtained from (108) by changing $\rho_\leftarrow$ to $1-\rho_\rightarrow$.

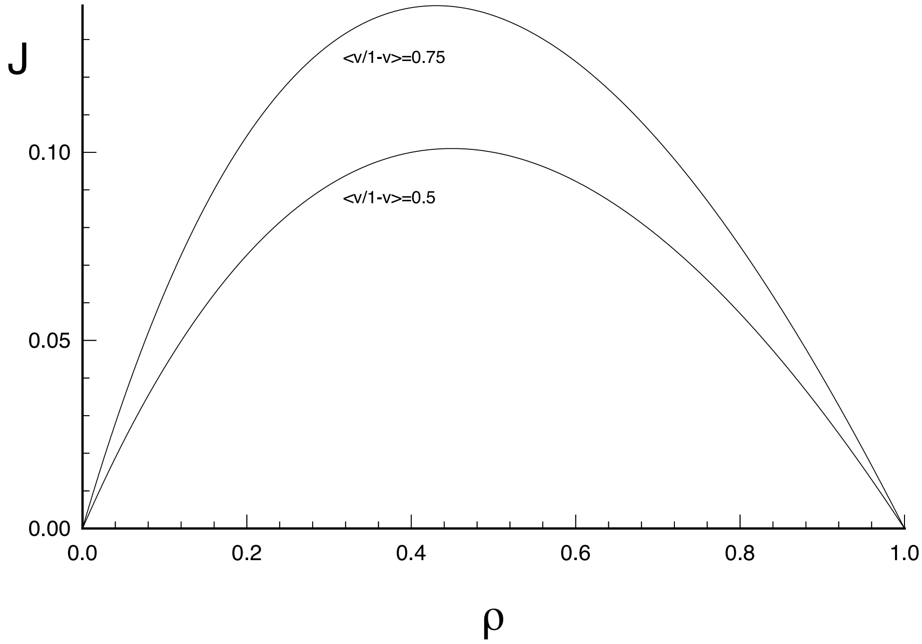

Figure 10. The current versus the density for different values of $\langle \frac{v}{1-v} \rangle$ in backward updating.

Figure 10 shows $J_{\rightarrow}$ versus $\rho_{\rightarrow}$ for different values of $\langle \frac{v}{1-v} \rangle$. The maximum of $J_{\rightarrow}$ has moved to the left. This maximum occurs at

$$\rho_{\rightarrow}^{max}\left(\left\langle \frac{v}{1-v} \right\rangle\right) = \frac{1}{\langle \frac{v}{1-v} \rangle}\left[\left(1+\left\langle \frac{v}{1-v} \right\rangle\right)^{\frac{1}{2}} - 1\right] \leqslant \frac{1}{2}. \tag{114}$$

7. Comparison and concluding remarks

Here we compare our results with those of [20] and specify the similarities and differences between ordered and random sequential updating procedures. We first discuss the similarities. Through the mapping procedure, the three types of update, i.e. random sequential (RS), backward sequential (BS) and forward sequential (FS), have proven to be described by quadratic algebras with similar structures. Rate (probability) of injection of particles is proportional to their velocities in all three schemes. Also, the extraction rate (probability) of a particle appears as a function of its velocity (see table 1). These dependences are consequences of the form of the quadratic algebras (5)–(8), (42)–(45). In all schemes, the steady current of each species is proportional to the total current. The proportionality constant is the hopping rate. Another feature which is common in the large-p limit is the sharp increase in the density of low-speed particles, which can somehow be interpreted as a kind of Bose–Einstein condensation (see equation (82)).

Now we discuss the differences of the schemes. When considering the infinite-species limit, one can investigate the characteristics of both schemes with a limited number of control parameters. As far as analytical calculations are concerned, these control parameters are α, e and m in ordered schemes and α, m and λ in the random scheme where m and λ determine the shape of distribution function [20] (see table 1). One of the advantages of the ordered scheme

Table 1. Comparison of the three updating schemes. RS: random sequential, BS: backward sequential, FS: forward sequential. Injection rate (α_i) and extraction rate (β_i) given in terms of hopping rate (v_i).

	RS	BS	FS
Injection rate (α_i)	$\frac{\alpha}{p} v_i$	$\frac{v_i}{p} \frac{\alpha}{e}$	$\frac{v_i}{p} \frac{\alpha}{e}$
Extraction rate (β_i)	$v_i + \bar{\beta} - 1$	$(1+\gamma)v_i - \gamma$	$(1+\gamma)v_i - \gamma$
Velocity distribution in large-p limit	$P(v) \sim (v-\alpha)^m \mathrm{e}^{\frac{-(v-\alpha)}{\lambda}}$	$P_1(v) \sim (v-\alpha)^m$ $P_2(v) \sim P_1(v)(1-v)$	$P_1(v) \sim (v-\alpha)^m$ $P_2(v) \sim P_1(v)(1-v)$
Control parameters	m, λ, α	m, e, α	m, e, α
Current of i-type particles ($J^{(i)}$)	$J^{(i)} = \frac{v_i}{p} J_{RSU}$	$J_{\leftarrow}^{(i)} = \frac{v_i}{p} J$	$J_{\rightarrow}^{(i)} = \frac{v_i}{p} J$
Mean field line	$(\alpha + \beta) = 1$	$(1-\alpha)(1-\bar{\beta}) = 1 - e$	$(1-\alpha)(1-\bar{\beta}) = 1 - e$

is the appearance of the more physical parameter e in control parameters, which is absent in the random scheme. Recalling that e = average hopping probability, in RS, time is rescaled such that e equals one. On the contrary, in ordered updating e remains as a free parameter. This is one of the main differences between two updating schemes.

Regarding BS and FS, one observes distinctive differences in their associated diagrams. Comparing figures 1 and 6, the left-shifting of the value of the density where the current is maximum is depicted. The main difference between figures 2 and 7 is the existence of non-vanishing current at zero density in figure 7. This is due to the forward nature of the updating which allows the created particle at the first site to move freely along the chain. Comparing figures 3 and 8, one does not observe any qualitative difference.

In this paper, we made a more complete investigation of the current–density and current diagrams for different regions of parameter space. We also evaluated the dependence of the current on the density for fixed values of α in RS. The corresponding diagram is very similar to figure 2, only the values of current and minimum allowed value of the density are different.

As demonstrated in the previous sections, setting $p = 1$, one recovers all the results obtained in the usual ASEP [23]. All the results of this paper and [20] have been obtained in a restricted region of parameters space (α_i, β_i, v_i) where mean field approximation becomes exact. It would be a highly nontrivial task to investigate the physical properties of the entire region of parameter space either by infinite-dimensional representations or by the explicit use of quadratic algebra.

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